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On a Conjecture by Herstein

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(1) Assume that R has no nonzero nil two-sided ideals. Let $k \geqslant 1$ be a fixed positive integer. The following two results are shown: (a) If, to each pair of elements x, y of R, there correspond positive integers $m = m(x, y)$, $n = n(x, y) \geqslant 1$ such that

$$\underbrace{[\cdots[[x^m, y^n], y^n]\cdots, y^n]}_{k\text{-times}} = 0,$$

then R is commutative. (b) If $a \in R$ is such that, to each y of R, there corresponds a positive integer $n = n(a, y) \geqslant 1$ such that

$$\underbrace{[\cdots[[a, y^n], y^n]\cdots, y^n]}_{k\text{-times}} = 0,$$

then a is in the center of R.

(2) Assume that R has no nonzero nil one-sided ideals. Then the same results as in (a) and (b) above are shown under the weaker assumptions that the k in (a) depends on the pair of elements $x, y \in R$ and that the k in (b) depends on both a and y. © 1989 Academic Press, Inc.

Since the late 1940s, shortly after the development of the general structure theory for associative rings, a great deal of work has been done which shows that certain types of hypotheses imply the commutativity of a ring. Among the results obtained, two of the most outstanding are the hypercenter theorem by Herstein [10] and Faith's conjecture by Anan'in and

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Zyabko [1] and by Herstein [11] independently. The aim of this paper is to generalize both of these results in the form conjectured in [14].

For an associative ring R, let $J(R)$, $N(R)$, and $P(R)$ stand for the Jacobson radical, the nil radical, and the prime radical of R, respectively. Also let $N_r(R)$ denote the sum of all nil right ideals of R. It is well known that $N_r(R)$ is also the sum of nil left ideals of R and that $N_r(R)$ is thus a two-sided ideal of R. Also, let $Z(R)$ denote the center of R.

For positive integers $k \geqslant 1$, we define $[x, y]_k$ for $x, y \in R$ inductively as follows: $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k \geqslant 2$.

DEFINITION. For each positive integer $k \geqslant 1$, a ring R is said to be a *C_k-ring* if the following condition is satisfied:

(C_k) For given $x, y \in R$, there exist positive integers $m = m(x, y) \geqslant 1$, $n = n(x, y) \geqslant 1$, depending on both x and y, such that $[x^m, y^n]_k = 0$.

DEFINITION. For each positive integer $k \geqslant 1$, the kth hypercenter $T_k(R)$ of R is $T_k(R) = \{a \in R$: for all $x \in R$, there exist integers $n = n(a, x) \geqslant 1$ such that $[a, x^n]_k = 0\}$.

Our main objective is to prove the following

THEOREM 1. *Assume that $N(R) = 0$. If R is a C_k-ring, then R is commutative.*

THEOREM 2. *Assume that $N(R) = 0$. Then $T_k(R) = Z(R)$.*

We remark that Theorem 1 was conjecture in [4, 14, 16, 18]. (There are many errors in [4]: cf. [5, 6, 7].) Note that the hypercenter $T(R)$ of [10] is simply the first hypercenter here. So Theorem 2 above generalizes the hypercenter theorem in [10].

As in [4], we introduce the following more general notions:

DEFINITION. A ring R is said to be a *C-ring* if the following condition (C) is satisfied.

(C) For given x, y in R, there exist positive integers $m = m(x, y) \geqslant 1$, $n = n(x, y) \geqslant 1$, and $k = k(x, y) \geqslant 1$ such that $[x^m, y^n]_k = 0$.

DEFINITION. The *generalized hypercenter* of R is the set

$$H(R) = \{a \in R \mid \text{for all } y \in R, \text{ there exist positive integers } n = n(a, y) \geqslant 1 \text{ and } k = k(a, y) \geqslant 1 \text{ such that } [a, y^n]_k = 0\}.$$

The following two conjectures, which were raised in Drazin [4] and which still remain open, are natural generalizations of Theorem 1 and Theorem 2, respectively.

Conjecture C. Let R be a C-ring. If $N(R)=0$, then R is commutative.

Conjecture H. If $N(R)=0$, then $H(R)=Z(R)$.

Our next objectives are the following two theorems:

THEOREM 3. *Assume that R is a C-ring. If $N_r(R)=0$, then R is commutative.*

THEOREM 4. *If $N_r(R)=0$, then $H(R)=Z(R)$.*

Theorem 3 and Theorem 4 might be known more or less to researchers in this area as folklore. However, the results in this generality do not seem to exist in any literature known to the authors. Their proofs are by no means trivial either: For the proof of Theorem 3, the argument at a crucial step involves a nontrivial application of a nontrivial theorem due to Chacron, Lawrence, and Madison [3]. The proof of Theorem 1 is essentially a modification of Herstein's proof of the hypercenter theorem [10]. However, there is a particular difficulty pertaining to rings with $N_r(R)=0$: Unlike rings with $N(R)=0$, we do *not* have a representation of a ring with $N_r(R)=0$ as a subdirect product of prime rings enjoying the same property. For these reasons, we have decided to include them there.

DEFINITION [4, p. 895]. A ring R is said to be a K-ring if the following condition (K) is satisfied:

(K) For given x, y in R, there exist positive integers $n=n(x,y)\geqslant 1$ and $k=k(x,y)\geqslant 1$ such that $[x, y^n]_k=0$.

A K-ring is trivially a C-ring. Also, it is obvious that R is a K-ring if and only if $H(R)=R$. The following conjecture is thus weaker than both Conjecture C and Conjecture H.

Conjecture K. Let R be a K-ring. If $N(R)=0$, then R is commutative.

Our last result, which generalizes [17], is the following

THEOREM 5. *Conjecture C, Conjecture H, and Conjecture K are equivalent to each other.*

For logical reasons, the material is organized as follows: The proof of Theorem 1 is given in the first section. Then the proofs of Theorem 4, Theorem 2, Theorem 3, and Theorem 5 are given in this order in the consecutive sections. We also single out two corollaries in Section 1 and Section 3, respectively, which might be of independent interest: The first corollary asserts that a torsion-free, semiprime C_k-ring, possessing a regular element, must be commutative. The second corollary says that, assuming $N(R)=0$, if to each pair of elements x, y of R, there correspond positive integers $k=k(x)\geqslant 1$, $m=m(x)\geqslant 1$, and $n=n(x, y)\geqslant 1$ such that $[x^m, y^n]_k=0$, then R must be commutative.

The following simple facts will be used implicitly throughout all the proofs of this paper: Let $x, y, z\in R$. If $[x, y]_k=0$ for some $k\geqslant 1$, then $[x, y^n]_k=0$ for any $n\geqslant 1$ and $[x, y]_l=0$ for any $l\geqslant k$. Also, if $[x, y^s]_k=0$ and $[z, y^t]_k=0$ for some $s\geqslant 1$, $t\geqslant 1$, then $[x, y^{st}]_k=[z, y^{st}]_k=0$.

1. The Proof of Theorem 1

Our proof of Theorem 1 is essentially based on Herstein [14]. For the sake of clarity and also for convenience of reference, we reorganize those well-known facts we will need later, mostly from Herstein [14], in Lemma 1 to Lemma 6. Some of them are also slightly sharpened for our purpose.

Lemma 1. *Let R be a C_k-ring such that $N(R)=0$. If R is torsion, then R is commutative.*

Proof. For each prime $p>0$, define $R_p=\{x\in R\colon px=0\}$. Since R is a semiprime torsion ring, R is a direct sum of R_p's. It suffices to prove the commutativity of each R_p. Observe that $N(R_p)\subseteq N(R)=\{0\}$. So we may assume that $R=R_p$ for some prime $p>0$ from the beginning. Pick $s\geqslant 1$ such that $p^s\geqslant k$. For $x, y\in R$ $(=R_p)$ and for $m, n\geqslant 1$ such that $[x^m, y^n]_k=0$, we immediately have that $0=[x^m, y^n]_k=[[x^m, y^n]_k, y^n]_{(p^s-k)}=[x^m, y^n]_{p^s}=[x^m, y^{np^s}]$. By the result of [1] or [11], R is commutative.

Lemma 2. *Suppose that R is a torsion-free ring. Let $l\geqslant 1$ and let $y\in R$ be such that, for each $x\in R$, there exist $m=m(x, y)$, $n=n(x, y)\geqslant 1$ such that $[x^m, y^n]_l=0$. Then there exist $m'=m'(x, y)$, $n'=n'(x, y)\geqslant 1$ such that $[x^{m'}, y^{n'}]_l=0$ and also $[x^{m'}, y^{n'}]_{(l-1)}$ is nilpotent.*

Proof. Let $m, n\geqslant 1$ be such that $[x^m, y^n]_l=0$. Using x^{2m} in place of x, there exist $s, t\geqslant 1$ such that $[x^{2ms}, y^t]_l=0$. Set $u=nt$. Then $[x^m, y^u]_l=[x^{2ms}, y^u]_l=0$. It suffices to show that $[x^m, y^u]_{(l-1)}$ is nilpotent. Expand $0=[x^{2ms}, y^u]_{2s(l-1)}=\mu([x^m, y^u]_{(l-1)})^{2s}$, where μ is a positive integer. Since R is torsion free, $([x^m, y^u]_{(l-1)})^{2s}=0$ as desired.

Lemma 3. *Let R be a C_k-ring. For $b\in R$, define $A(b)=\{r\in R\colon rb^t=0$ for* *some integer* $t \geqslant 1\}$ *and* $B(b) = \{r \in R: b^t r = 0$ *for some integer* $t \geqslant 1\}$. *Then* $A(b) = B(b)$ *is a two-sided ideal of* R.

Proof. If $u \in A$, then $ub^t = 0$ for some $t \geqslant 1$ and hence also $(b^t u)^2 = 0$. By the C_k-condition on R, there exist $m, n \geqslant 1$ such that $0 = [(b^t + b^t u)^m, b^n]_k = [b^{mt} + b^{mt}u, b^n]_k = [b^{mt}u, b^n]_k$. We may assume that $n \geqslant t$. Then $[b^{mt}u, b^n]_k = (-1)^k b^{nk+mt}u = 0$. So $u \in B(b)$ and hence $A(b) \subseteq B(b)$. Similarly, we can show $B(b) \subseteq A(b)$. Since $A(b)$ and $B(b)$ are respectively a left ideal and a right ideal of R, $A(b) = B(b)$ is a two-sided ideal of R.

LEMMA 4 (Neumann [19]). *Let* R *be a* C_k*-ring. If* $N_r(R) = 0$, *then* R *is commutative.*

Proof. Let $a \in R$ be such that $a^2 = 0$. For $x \in R$, we have $a \in A(ax) = B(ax)$ by Lemma 3. So there exists $r \geqslant 1$ such that $(ax)^r a = 0$. This says that aR is nil. By the assumption $N_r(R) = 0$, $a = 0$. Thus R does not have nonzero nilpotent elements. By the result of [2], R is a subdirect product of domains R_α. It suffices to prove the commutativity of each R_α. So we may assume from the start that R is a domain. If char $R = p > 0$, the result follows from Lemma 1. So assume that char $R = 0$. By Lemma 2, given $x, y \in R$, there exist $m, n \geqslant 1$ such that $[x^m, y^n]_{k-1}$ is nilpotent and hence such that $[x^m, y^n]_{k-1} = 0$. Thus R is a C_{k-1} ring. Continuing in this manner, we can show finally that R is a C_1-ring. Now the lemma follows from [1] or [11].

Here we would like to remark that the proof of Lemma 4 [14] does not quite prove our Lemma 4 above, for there R has been reduced via a subdirect product and it is hard to see that such a reduction can preserve the condition $N_r(R) = 0$.

LEMMA 5 (Neumann [19]). *Suppose that* R *is a torsion-free* C_k*-ring. Assume that* R *has the unit element* 1. *Then* (1) *for any nilpotent elements* $a, b \in R$, $[a, b]_k = 0$, *and* (2) *the set of all nilpotent elements of* R *forms an ideal of* R, *namely the prime radical* $P(R)$ *of* R.

Proof. We claim that for any nilpotent element $a \in R$ and for any $y \in R$, there exists $n \geqslant 1$ such that $[a, y^n]_k = 0$. We proceed by induction on the index of nilpotency of a. If $a^2 = 0$, then, by the condition C_k, there exist $m, n \geqslant 1$ such that $0 = [(1+a)^m, y^n]_k = [1 + ma, y^n]_k = m[a, y^n]_k$. Since R is torsion-free, $[a, y^n]_k = 0$. Now suppose that $a^{s+1} = 0$, but $a^s \neq 0$ for some $s \geqslant 2$. By the induction hypothesis, pick $r \geqslant 1$ such that $0 = [a^2, y^r]_k = [a^3, y^r]_k = \cdots = [a^{s-1}, y^r]_k = [a^s, y^r]_k$. There exist $m, n \geqslant 1$ such that $[(1+a)^m, y^{rn}]_k = 0$. Using the fact that $[a^2, y^{rn}]_k = \cdots = [a^s, y^{rn}]_k = 0$, we compute $0 = [(1+a)^m, y^{rn}]_k = [1 + \binom{m}{1}a + \cdots + \binom{m}{s}a^s, y^{rn}]_k = \binom{m}{1}[a, y^{rn}]_k$. Since R is torsion-free, $[a, y^{rn}]_k = 0$ as desired.

Now let $a, b \in R$ be nilpotent. Assume that $b^{s+1}=0$ but $b^s \neq 0$ for some $s \geqslant 1$. Consider $1+\lambda b$ for $\lambda = 1, 2, ..., ks$. By the claim above, there exists $n \geqslant s$ such that $[a, (1+\lambda b)^n]_k = 0$ for $\lambda = 1, 2, ..., ks$. Expand $0 = [a, (1+\lambda b)^n]_k = [a, 1 + \binom{n}{1}(\lambda b) + \binom{n}{2}(\lambda b)^2 + \cdots + \binom{n}{s}(\lambda b)^s]_k = \lambda^k \binom{n}{1}^k [a, b]_k + \binom{k}{1} \lambda^{k+1} \binom{n}{1}^{k-1} \binom{n}{2} [[a, b^2], b]_{k-1} + \cdots + \binom{k}{k} \lambda^{ks} \binom{n}{s}^k [a, b^s]_k$. Using the Vandermonde determinant, we can solve $[a, b]_k = 0$. This proves (1).

For (2), let $P(R)$ be the prime radical of R. Obviously, $R/P(R)$ is a C_k-ring with a unit element. Also, note that $R/P(R)$ is torsion-free (see Appendix for a proof). Replacing R by $R/P(R)$, we may assume that R is semiprime from the start. Since a commutative semiprime ring does *not* possess nonzero nilpotent elements, it suffices to show that R is commutative. If R has a nonzero nil right ideal, say $\rho \neq 0$, then ρ satisfies the polynomial identity $[x, y]_k = 0$ by (1) of this lemma. But then Lemma 2.1.1 [12] says that R cannot be semiprime. This is absurd. Hence $N_r(R) = 0$. Now the commutativity of R follows from Lemma 4.

Let y be a regular element of R. As in [14], we define $C(y) = \{r \in R: ry = yr\}$, $W(y) = \{r \in R: ry^t = y^t r$ for some integer $t \geqslant 1\}$ and $N(y)$ to be the set of nilpotent elements of $W(y)$.

LEMMA 6. *Suppose that R is a torsion-free C_k-ring. Let y be regular in R. Then* (1) *$N(y)$ is the prime radical of $W(y)$ and* (2) *$N(y)$ satisfies the polynomial identity $[u, v]_k = 0$.*

Proof. Let $a, b \in N(y)$ and $x, z \in W(y)$. Choose $r \geqslant 1$ such that $a, b, x, z \in C(y^r)$. Let D be the localization of $C(y^r)$ at the semigroup generated by y^r. Then D is obviously a C_k-ring with the unit element 1. Applying Lemma 5 to D, $a+b$, ax, za, and $[x, z]$ are all nilpotent in D and hence all of them belong to $N(y)$. Also $[a, b]_k = 0$. Hence $N(y)$ really forms an ideal of $W(y)$ and satisfies the polynomial identity $[u, v]_k = 0$. Let P be the prime radical of $W(y)$. Obviously, $N(y) \supseteq P$. If $N(y) \not\subseteq P$, then $N(y)/P$ would be a nonzero nil ideal of $W(y)/P$ satisfying a polynomial identity. By Lemma 2.1.1 [12], $W(y)/P$ cannot be semiprime. This is absurd. So $P = N(y)$ as desired.

Our method for proving Theorem 1 is to improve Lemma 5 by weakening the assumption on the existence of the unit element. In Lemma 5, observe that, assuming the existence of the unit element, the commutativity of R follows from $P(R) = 0$ instead of $N(R) = 0$. This is really crucial in making the following

DEFINITION. Let $l \geqslant 1$ be a positive integer. A ring R is said to be l-good if R is a torsion-free, semiprime C_k-ring which possesses a regular element y satisfying the following condition:

(G_l) For each $x \in R$, there exist positive integers $m = m(x, y) \geqslant 1$, $n = n(x, y) \geqslant 1$ such that $[x^m, y^n]_l = 0$.

We emphasize that, instead of $N(R) = 0$, $P(R) = 0$ is assumed in the above definition. It is the deletion of the assumption $N(R) = 0$ that enables us to do some sort of induction in the following lemmas.

LEMMA 7. *If R is 1-good, then R is commutative.*

Proof. Let y be a regular element in R satisfying the condition (G_1). We claim that $N(y) \supseteq N(x)$ for every regular element x in R. Let $a \in N(x)$ be such that $a^{u+1} = 0$ and $a^u \neq 0$, where $u \geqslant 1$. Using the condition (G_1) and the definition of $N(x)$, there exist $s, t \geqslant 1$ such that $[a, x^s] = 0$ and $[x^s, y^t] = 0$. Consider $x^s + \lambda a$ for $\lambda = 1, ..., u$. By the condition (G_1) again, there exist $m \geqslant u \geqslant 1$, $n \geqslant 1$ such that $[(x^s + \lambda a)^m, (y^t)^n] = 0$ for $\lambda = 1, ..., u$. Compute

$$\begin{aligned} 0 &= [(x^s + \lambda a)^m, y^{tn}] \\ &= \left[x^{ms} + \binom{m}{1} x^{(m-1)s}(\lambda a) + \cdots + \binom{m}{u} x^{(m-u)s}(\lambda a)^u, y^{tn}\right] \\ &= \lambda \binom{m}{1} x^{(m-1)s}[a, y^{tn}] + \cdots + \lambda^u \binom{m}{u} x^{(m-u)s}[a^u, y^{tn}]. \end{aligned}$$

Using the Vandermonde argument, we have $\binom{m}{1} x^{(m-1)s}[a, y^{nt}] = 0$ and hence $[a, y^{nt}] = 0$. So $a \in N(y)$ as desired.

We have thus shown that $N(y) \supseteq N(x)$ for any regular element x in R. Thus $N(y)$ is the unique maximal element of the family $\{N(x): x$ is regular in $R\}$ with respect to the inclusion. This is an invariant property. Thus $(1-a)\, N(y)(1-a)^{-1} \subseteq N(y)$ for any nilpotent element $a \in R$. Let $b \in N(y)$ and let $a \in R$ be such that $a^{u+1} = 0$ and $a^u \neq 0$, where $u \geqslant 1$. For $\lambda = 1, ..., u$,

$$\begin{aligned} (1-\lambda a)\, b(1-\lambda a)^{-1} &= (1-\lambda a)\, b(1 + (\lambda a) + \cdots + (\lambda a)^u) \\ &= b + \lambda(ba - ab) + \lambda^2(ba^2 - aba) + \cdots \\ &\quad + \lambda^{u+1}(-aba^u) \in N(y). \end{aligned}$$

Using the Vandermonde argument again, we can solve $\mu[a, b] \in N(y)$, where μ is a nonzero integer. Since R is torsion-free, $[a, b] \in N(y)$. We have thus shown that $[a, N(y)] \subseteq N(y)$ for any nilpotent element $a \in R$. Since any element of $N_r(R)$ is a sum of nilpotent elements, we have $[N_r(R), N(y)] \subseteq N(y)$. If $N_r(R) = 0$, then R is commutative by Lemma 4 and there is nothing to prove. So we assume that $N_r(R) \neq 0$. Let $a, b \in N(y)$. Compute $N(y) \supseteq [a, bN_r(R)] = [a, b]\, N_r(R) + b[a, N_r(R)]$.

Since $b[a, N_r(R)] \subseteq bN(y) \subseteq N(y)$, we have $[a, b]\, N_r(R) \subseteq N(y)$. Hence $[a, b]\, N_r(R)$ is a nil right ideal of R satisfying the polynomial identity $[u, v]_k = 0$. Since R is semiprime, $[a, b]\, N_r(R) = 0$ by Lemma 2.1.1 [12]. So we have $[N(y), N(y)]\, N_r(R) = 0$. Let $a \in N(y)$. Then $[a, N_r(R)] \subseteq N(y)$ and hence $[a, [a, N_r(R)]]\, N_r(R) = 0$. Since $[a, [a, N_r(R)]] \subseteq N_r(R)$ and since $N_r(R)$ is semiprime, $[a, [a, N_r(R)]] = 0$. Let $x, z \in N_r(R)$. Then $0 = [a, [a, xz]] = [a, [a, x]z + x[a, z]] = 2[a, x][a, z]$. Substitute zx for z in the above formula and expand $0 = 2[a, x][a, zx] = 2[a, x]([a, z]x + z[a, x]) = 2[a, x]\, z[a, x]$. Thus $[a, x]\, N_r(R)[a, x] = 0$. Since $[a, x] \in N_r(R)$, $[a, x] = 0$ by the semiprimeness of $N_r(R)$. We have thus shown $[N(y), N_r(R)] = 0$. In particular, $N(y) \cap N_r(R)$ is contained in the center of $N_r(R)$. Since the center of a semiprime ring does *not* have nonzero nilpotent elements, we have $N(y) \cap N_r(R) = \{0\}$.

Let $a \in N_r(R)$ be such that $a^2 = 0$. By the condition (G_1) for R, there exists $m \geqslant 1$ such that $[(1+a)\, y^m(1-a), y^m] = 0$ and also $[(1-a)\, y^m(1+a), y^m] = 0$. Then $2[[a, y^m], y^m] = [(1+a)\, y^m(1-a) - (1-a)\, y^m(1+a), y^m] = 0$. Since R is torsion-free, $[a, y^m] \in W(y)$. Also, $0 = [[a^2, y^m], y^m] = 2[a, y^m][a, y^m]$. Thus $[a, y^m] \in N(y)$ and, since $a \in N_r(R)$, $[a, y^m] \in N(y) \cap N_r(R) = \{0\}$. So we have $a \in W(y)$. Since $a^2 = 0$, $a \in N(y)$ and hence $a \in N(y) \cap N_r(R) = \{0\}$. We have thus shown that there are *no* nonzero elements of square 0 in $N_r(R)$. Thus $N_r(R) = 0$, absurd! The lemma is thus proved.

Let R be a C_k-ring. If $y \in R$ is regular, we define $T(y) = \{r \in N(y): rN(y) = 0\}$.

LEMMA 8. *Let $l > 1$ be such that every $(l-1)$-good ring is commutative. Suppose that R is a noncommutative l-good ring. Let y be a regular element in R satisfying the condition (G_l) for R. Then $T(y) = 0$.*

Proof. Assume towards a contradiction that $T(y) \neq 0$. Let $0 \neq a \in T(y)$. Then $a^2 = 0$. Pick $r \geqslant 1$ such that $[a, y^r] = 0$. Set $z = y^r$. Let S be the subring of R generated by z and Ra. Every element s of S can be written in the form $s = p(z) + xa$, where $x \in R$ and $p(z)$ is a polynomial in z with integer coefficients.

Let $r(a) = \{a \in R: ax = 0\}$, the right annihilator of a in R. $r(A) \cap S$ is obviously a right ideal of S. For $s = p(z) + xa \in S$, where $x \in R$ and $p(z)$ is a polynomial in z with integer coefficients, we have $as(r(a) \cap S) = a(p(z) + xa)(r(a) \cap S) = ap(z)(r(a) \cap S) = p(z)\, a(r(a) \cap S) = 0$. Hence $s(r(a) \cap S) \subseteq r(a) \cap S$. We have thus shown that $r(a) \cap S$ is actually a two-sided ideal of S.

Now consider the ring $\bar{S} = S/r(a) \cap S$. We claim that $\bar{S}$ is $(l-1)$-good. Let $s \in S$ be such that $\bar{s}\bar{S}\bar{s} = 0$. Then $asSs = 0$ and, in particular, $asRas = 0$. Since R is semiprime, we have $as = 0$ and hence $\bar{s} = 0$ in $\bar{S}$. So $\bar{S}$ is semiprime. Let $s \in S$ be such that $n\bar{s}=0$ for some positive integer $n \geqslant 1$. Then $nas=0$. Since R is torsion-free, $as=0$ and hence $\bar{s}=0$. So $\bar{S}$ is torsion-free. It is obvious that $\bar{S}$ also satisfies the condition (C_k). Thus $\bar{S}$ is a torsion-free, semiprime C_k-ring. Now we show that $\bar{z}$ is a regular element in $\bar{S}$ satisfying the condition (G_{l-1}) for $\bar{S}$. Let $s \in S$. If $\bar{z}\bar{s}=0$, then $0=a(zs)=z(as)$. Since z is regular in R, $as=0$ and hence $\bar{s}=0$. Similarly, if $\bar{s}\bar{z}=0$, then $asz=0$. By the regularity of z, $as=0$. Hence we also have $\bar{s}=0$. Thus $\bar{z}$ is regular in $\bar{S}$. We verify the condition (G_{l-1}) on $\bar{S}$. Let $s=p(z)+xa \in S$. Since R is l-good, by Lemma 2, there exist $m, n \geqslant 1$ such that $[s^m, z^n]_{(l-1)} \in N(z) \subseteq N(y)$. Then by the fact $a \in T(y)$, $a[s^m, z^n]_{(l-1)}=0$. So $[\bar{s}^m, \bar{z}^n]_{(l-1)}=0$. We have thus shown that $\bar{S}$ is $(l-1)$-good as claimed.

By our assumption on l, $\bar{S}$ is commutative. In particular, $[\bar{u}\bar{a}, \bar{v}\bar{a}]=0$ for all $u, v \in R$. That is, $auava-avaua=0$ for $u, v \in R$. Let P be a prime ideal of R. If $a \notin P$, then $\tilde{R}=R/P$ satisfies the nontrivial GPI. $\tilde{a}\tilde{u}\tilde{a}\tilde{v}\tilde{a}-\tilde{a}\tilde{v}\tilde{a}\tilde{u}\tilde{a}=0$. By a result of Jain (Corollary 7.5.15, p. 280 [20]), $N_r(R/P)=0$. By Lemma 4, R/P is commutative. Since there are *no* nonzero nilpotent elements in a commutative prime ring, $\tilde{a}=0$ and hence $a \in P$, a contradiction. So a belongs to every prime ideal of R. Since R is semiprime, the intersection of all prime ideals of R is $\{0\}$. So $a=0$, a contradiction again.

LEMMA 9. *Let $l>1$ be such that every $(l-1)$-good ring is commutative. Suppose that R is a noncommutative l-good ring. Let y be a regular element in R satisfying the condition (G_l) on R. Then for each $b \in N(y)$, $b^k=0$.*

Proof. Suppose not. Let $b \in N(y)$ be such that $b^k \neq 0$. Assume that $b^q \neq 0$ but $b^{q+1}=0$. Then $k \leqslant q$. Pick $r \geqslant 1$ such that $[b, y^r]=0$. Set $a=b^q$ and $z=y^r$. Then $[a, z]=[b, z]=0$. Let S be the subring of R generated by z and Ra. Let $r(a)$ be the right annihilator of a in R. As in the proof of Lemma 8, $r(a) \cap S$ is a two-sided ideal of S. Consider the ring $\bar{S}=S/r(a) \cap S$. As in the proof of Lemma 8, $\bar{S}$ is a torsion-free, semiprime C_k-ring and $\bar{z}$ is regular in $\bar{S}$. We show that $\bar{z}$ satisfies the condition (G_{l-1}) on $\bar{S}$.

Let $s \in S$. Since R is l-good, by Lemma 2, there exist $m, n \geqslant 1$ such that $[s^m, z^n]_{(l-1)} \in N(z) \subseteq N(y)$. By Lemma 6, $[[s^m, z^n]_{(l-1)}, b]_k=0$. Since $q \geqslant k$, we have $[[s^m, z^n]_{(l-1)}, b]_q=0$. As in the proof of Lemma 8, we write $s=p(z)+xa$, where $x \in R$ and $p(z)$ is a polynomial in z with integer coefficients. Using the fact $ab=b^q \cdot b=b^{q+1}=0$, we compute $sb=(p(z)+xa)b=p(z)b=bp(z)$. Hence $s^m b=s^{m-1}(sb)=s^{m-1}(bp(z))=s^{m-2}(b(p(z))^2)=\cdots=b(p(z))^m$. Using this, $[s^m, b]=s^m b-bs^m=b(p(z))^m-bs^m$. Also, $[[s^m, b], z^n]_{(l-1)}=[s^m b-bs^m, z^n]_{(l-1)}=[b(p(z))^m-bs^m, z^n]_{(l-1)}=[-bs^m, z^n]_{(l-1)}=(-b)[s^m, z^n]_{(l-1)}$, by the fact $[z, b]=0$. Now, using the identity obtained above, we compute

$$0 = [[s^m, z^n]_{(l-1)}, b]_q = [[[s^m, b], z^n]_{(l-1)}, b]_{q-1}$$
$$= [(-b)[s^m, z^n]_{(l-1)}, b]_{(q-1)} = (-b)[[s^m, z^n]_{(l-1)}, b]_{(q-1)}$$
$$= (-b)[[[s^m, b], z^n]_{(l-1)}, b]_{(q-2)} = (-b)^2 [[s^m, z^n]_{(l-1)}, b]_{(q-2)}$$
$$= \cdots = (-b)^q [s^m, z^n]_{(l-1)} = (-1)^q a[s^m, z^n]_{(l-1)}.$$

So $[s^m, z^n]_{(l-1)} \in r(a) \cap S$ and hence $[\bar{s}^m, \bar{z}^n]_{(l-1)} = 0$. Thus $\bar{z}$ satisfies the condition (G_{l-1}) on $\bar{S}$. So $\bar{S}$ is $(l-1)$-good. By our assumption on l, $\bar{S}$ is commutative. In particular $[\overline{ua}, \overline{va}] = 0$ for $u, v \in R$ or equivalently, $auava - avaua = 0$. Arguing as in the proof of Lemma 8, we have $a = 0$ again. This is a contradiction. The lemma is thus proved.

LEMMA 10. *Let $l \geqslant 1$. Then any l-good ring is commutative.*

Proof. Suppose not. Let $l \geqslant 1$ be the minimal integer such that there exists a noncommutative l-good ring R. By Lemma 7, $l > 1$. By the minimality of l, every $(l-1)$-good ring is commutative. Thus Lemma 8 and Lemma 9 are applicable to R. Let y be a regular element in R satisfying the condition (G_l) on R. Assume that $N(y) \neq 0$. By Lemma 9, $N(y)$ is nil of bounded index. By the Nagata–Higman theorem [15, p. 274], $N(y)$ is nilpotent and hence $T(y) \neq 0$, in contradiction with Lemma 8. So we must assume that $N(y) = 0$. By Lemma 2, for given $x \in R$, there exist $m, n \geqslant 1$ such that $[x^m, y^n]_{(l-1)} \in N(y) = \{0\}$. Then R is $(l-1)$-good and hence is commutative. This is a contradiction.

We state the following immediate consequence which might be of independent interest.

COROLLARY 1. *Let R be a torsion-free, semiprime C_k-ring. If R has a regular element, then R is commutative.*

Proof. If R has a regular element, then R is k-good. The result follows immediately from Lemma 10.

Now we are ready to give the proof of Theorem 1. It consists mainly of a reduction of a C_k-ring with $N(R) = 0$ to a subdirect product of prime rings with regular elements. This is due to Herstein [14]. For the sake of completeness, we also reproduce it here.

Proof of Theorem 1. Let R be a C_k-ring such that $N(R) = 0$. Using the embedding in Lemma 2.2.3 [9], R is a subdirect product of *strongly prime* rings R_α in the sense that each R_α possesses a nonnilpotent element b_α such that each nonzero ideal of R_α contains some power of b_α. It suffices to prove the commutativity of each R_α. Replacing R by R_α, we may assume from the start that R is strongly prime with a nonnilpotent element b such that any nonzero ideal of R contains some power of b. Let $A(b)$ and $B(b)$ be as defined in Lemma 3. If $A(b) \neq 0$, then, by Lemma 3, $A(b)$ is a nonzero ideal of R. Hence $b^s \in A(b)$ for some $s \geqslant 1$. By the definition of $A(b)$, $b^s b^t = 0$ for some $t \geqslant 1$. But then b is nilpotent, which is absurd. So we have $A(b) = B(b) = \{0\}$. Thus b is regular. If the characteristic of R is some prime $p > 0$, then R is commutative by Lemma 1. If the characteristic of R is 0, by Corollary 1 above, R is also commutative. The theorem is thus proved.

2. The Proof of Theorem 4

The key observation is that $H(R)$ forms a subring of R which is invariant under any automorphisms of R. Also note that $Z(R) \subseteq T_k(R) \subseteq H(R)$ for each integer $k \geqslant 1$.

We begin the proof with the following well-known result, whose proof is similar to that of Lemma 1.

Lemma 11. *If R is a prime ring of characteristic $p > 0$ and $N(R) = 0$, then $H(R) = Z(R)$.*

Proof. Let $a \in H(R)$. Given $y \in R$, there exist positive integers $k = k(a, y) \geqslant 1$ and $n = n(a, y) \geqslant 1$ such that $[a, y^n]_k = 0$. Pick $s \geqslant 1$ such that $p^s \geqslant k$. Then $0 = [[a, y^n]_k, y^n]_{(p^s - k)} = [a, y^n]_{p^s} = [a, y^{np^s}]$. Thus $a \in T_1(R)$ and hence $a \in Z(R)$ by the result of [10]. This proves that $H(R) \subseteq Z(R)$ and so $H(R) = Z(R)$.

Lemma 12. *Assume that R is a domain. Let $a, b \in H(R)$ be such that $a + b + ab = a + b + ba = 0$. Then $a, b \in Z(R)$.*

Proof. By Lemma 11, we may assume that $\operatorname{char} R = 0$. Given $y \in R$, let $k \geqslant 1$, $l \geqslant 1$ be the minimal integers such that $[a, y^n]_k = 0$ and $[b, y^n]_l = 0$ for some integer $n \geqslant 1$. Suppose that $k > 1$ and $l > 1$, then $k + l - 2 \geqslant k$. Hence $[a, y^n]_{(k+l-2)} = [b, y^n]_{(k+l-2)} = 0$. Compute $0 = [a + b + ab, y^n]_{(k+l-2)} = [ab, y^n]_{(k+l-2)} = \binom{k+l-2}{k-1}[a, y^n]_{(k-1)}[b, y^n]_{(l-1)}$. Since R is a domain of characteristic 0, we have $[a, y^n]_{(k-1)} = 0$ or $[b, y^n]_{(l-1)} = 0$. This contradicts with the minimality of k and l. Hence one of k and l must be 1, say $k = 1$. Then $0 = [a + b + ab, y^n] = [b, y^n] + [ab, y^n] = [b, y^n] + a[b, y^n] = (1 + a)[b, y^n]$ (note that the use of the unit 1 is purely formal). Hence $[b, y^n] = (1 + b)(1 + a)[b, y^n] = 0$. Thus $k = l = 1$. Since this holds for any $y \in R$, $a, b \in T_1(R)$. Hence $a, b \in Z(R)$ by the result of [10].

As an immediate consequence, we have

LEMMA 13. *Assume that R is a division ring. If $H(R)=R$ then R is commutative.*

Proof. Let $-1 \neq a \in R$. Set $b=(1+a)^{-1}-1$. Then $a+b+ab=a+b+ba=0$. By Lemma 12, $a \in Z(R)$. So R is commutative.

LEMMA 14. *Assume that R is a domain and $J(R) \neq 0$. If $H(R)=R$ then R is commutative.*

Proof. By Lemma 12, $J(R) \subseteq Z(R)$. Hence $J(R)R \subseteq J(R) \subseteq Z(R)$. Then $0=[J(R), R]=J(R)[R, R]$. Since $J(R) \neq 0$, $[R, R]=0$. Thus R is commutative.

LEMMA 15. *Let D be a division ring and let V be a left vector space over D. Suppose that R is a dense subring of* $\mathrm{Hom}({}_D V, {}_D V)$. *Then $H(R)=Z(R)$ unless* $\dim_D V=1$.

Proof. Let $a \in H(R)$ and $0 \neq v \in V$. We claim that v and va are D-dependent. Suppose not. Then, by the density of R, there exists $y \in R$ such that $vy=0$ and $vay=va$. For this y, there exist positive integers $n=n(a, y) \geqslant 1$, $k=k(a, y) \geqslant 1$ such that $[a, y^n]_k=0$. Then

$$0=v[a, y^n]_k=v\left(ay^{kn}-\binom{k}{1} y^n a y^{(k-1)n}+\cdots\right)=va.$$

This is a contradiction. Hence v and va are D-dependent.

Assume that $\dim_D V>1$. Let u, v be two independent vectors. By the claim above, there exist $\alpha, \beta, \gamma \in D$ such that $ua=\alpha u$, $va=\beta v$, and $(u+v)a=\gamma(u+v)$. But $(u+v)a=ua+va=\alpha u+\beta v$. Since u, v are independent, we have $\alpha=\beta=\gamma$. Now let w be an arbitrary nonzero vector in V. If u, w are independent then, by the above argument, $wa=\alpha w$. If u, w are dependent, then v, w are independent. By the above argument applying to the pair v, w, we also have $w=\alpha w$. Thus we have shown that there is $\alpha \in D$ such that $va=\alpha v$ for any $v \in V$. Let $\delta \in D$. Then $(\delta v)a=\alpha(\delta v)$. Also $(\delta v)a=\delta(va)=\delta\alpha a$. Hence $\alpha\,\delta v=\delta\alpha v$. So $\alpha\delta=\delta\alpha$ for all $\delta \in D$. Thus $\alpha \in Z(D)$ and hence $a \in Z(R)$ as desired.

LEMMA 16. *Assume that $J(R)=0$. If $H(R)=R$, then R is commutative.*

Proof. Via subdirect products, we may assume that R is a dense subring of linear transformations on a vector space V over a division ring D. If $\dim_D V=1$, then R is isomorphic to the division ring D and must be commutative by Lemma 13. If $\dim_D V>1$, then by Lemma 15, $R=H(R)=Z(R)$. So R is also commutative.

LEMMA 17. *Assume that R is a domain. If $H(R)=R$, then R is commutative.*

Proof. If $J(R)=0$ then the result follows from Lemma 16. If $J(R)\neq 0$ then our lemma is a consequence of Lemma 14.

A ring R is said to be *reduced* if R has no nonzero nilpotent elements. By the result of [2], a reduced ring is a subdirect product of domains. An immediate consequence of Lemma 17 is the following

LEMMA 18. *Assume that R is reduced. If $H(R)=R$, then R is commutative.*

LEMMA 19. *Assume that R is a division ring. Then $H(R)=Z(R)$.*

Proof. It suffices to show $[H(R), R]=0$. Suppose not. Let $a\in H(R)$ and $b\in R$ be such that $[a, b]\neq 0$. Then $b\neq 0, -1$. Set $a_1=bab^{-1}\in H(R)$ and $a_2=(1+b)\,a(1+b)^{-1}\in H(R)$. Then $ba=a_1b$ and $(1+b)a=a_2(1+b)$. Hence

$$a=(1+b)a-ba=a_2(1+b)-a_1b=a_2+(a_2-a_1)b. \qquad (*)$$

Since $H(H(R))=H(R)$ and $H(R)$ is obviously a domain, $H(R)$ is commutative by Lemma 17. Commuting $(*)$ with a and using the commutativity of $H(R)$, we get $(a_2-a_1)[a, b]=0$. If $a_2\neq a_1$, then $[a, b]=0$, a contradiction. So $a_2=a_1$. Substituting $a_2=a_1$ into $(*)$, we get $a=a_2$. Hence $(1+b)a=a(1+b)$. Thus $ab=ba$, a contradiction again. So we have $[a, b]=0$ for $a\in H(R)$ and $b\in R$.

Let D, V, R be as explained in Lemma 5. Combining Lemma 15 and Lemma 19, we have $H(R)=Z(R)$. As in Lemma 16, this result can be extended to the semiprimitive case in the following

LEMMA 20. *Assume that $J(R)=0$. Then $H(R)=Z(R)$.*

LEMMA 21. *Assume that $N_r(R)=0$. Then $H(R)$ is a reduced commutative subring of R.*

Proof. Assume that $a\in H(R)$ is such that $a^2=0$. For given $y\in R$, there exist positive integers $k=k(a, y)\geqslant 1$ and $n=n(a, y)\geqslant 1$ such that $[a, (a+ay)^n]_k=0$. Expanding $(a+ay)^n$ using the fact that $a^2=0$, we have $(a+ay)^n=(ay)^n+(ay)^{n-1}a$. Compute $[a, (a+ay)^n]=[a, (ay)^n+(ay)^{n-1}a]=[a, (ay)^n]=-(ay)^n a$. Hence $[a, (a+ay)^n]_k=(-1)^k(ay)^{nk}a$. Thus aR is a nil right ideal of R. Since $N_r(R)=0$, $a=0$ follows. So $H(R)$ is reduced. Since $H(H(R))=H(R)$, $H(R)$ is commutative by Lemma 18.

LEMMA 22. *Assume that $N_r(R)=0$. If $a \in H(R)$, then $2[a, J(R)]=0$.*

Proof. Let $a \in H(R)$ and $x \in J(R)$. Set $a_1=(1+x)\,a(1+x)^{-1}$ and $a_2=(1+2x)\,a(1+2x)^{-1}$. Since $H(R)$ is invariant under all automorphisms of R, a_1 and a_2 are in $H(R)$. Write $(1+x)a=a_1(1+x)$ and $(1+2x)a=a_2(1+2x)$. Compute $a=2(1+x)a-(1+2x)a=2a_1(1+x)-a_2(1+2x)=(2a_1-a_2)+2(a_1-a_2)x$. Commuting this last equality relation with a and using the commutativity of $H(R)$ by Lemma 21, we have $2(a_1-a_2)[a, x]=0$. We notice that

$$\begin{aligned} a_1-a_2 &= (1+x)\,a(1+x)^{-1}-(1+2x)\,a(1+2x)^{-1} \\ &= [(1+x)\,a(1+2x)-(1+2x)\,a(1+x)](1+x)^{-1}\,(1+2x)^{-1} \\ &= [a, x](1+x)^{-1}\,(1+2x)^{-1}. \end{aligned}$$

Then

$$2(a_1-a_2)^2=2(a_1-a_2)[a, x](1+x)^{-1}\,(1+2x)^{-1}=0.$$

Since $H(R)$ is reduced by Lemma 21, $2(a_1-a_2)=0$, so $2[a, x](1+x)^{-1}$ $(1+2x)^{-1}=0$. Thus $2[a, x]=0$ as desired.

Now we are ready to give

Proof of Theorem 4. Since $N_r(R)=0$, it is trivial that $N(R)=0$. So R is a subdirect product of prime rings R_α with $N(R_\alpha)=0$. Let π_α be the natural homomorphism of R onto R_α. It suffices to prove that $\pi_\alpha(H(R)) \subseteq Z(R_\alpha)$ for each α.

If char $R_\alpha=p>0$, then $H(R_\alpha)=Z(R_\alpha)$ by Lemma 11. But $\pi_\alpha(H(R)) \subseteq H(R_\alpha)$. So we may assume char $R_\alpha=0$. By Lemma 22, $2[H(R), J(R)]=0$. So $2[\pi_\alpha(H(R)), \pi_\alpha(J(R))]=0$ and hence $[\pi_\alpha(H(R)), \pi_\alpha(J(R))]=0$. If $\pi_\alpha(J(R)) \neq 0$, then $\pi_\alpha(H(R))$, which commutes with a nonzero ideal $\pi_\alpha(J(R))$ of the prime ring R_α, must be central in R_α. So we may also assume that $\pi_\alpha(J(R))=0$. But Lemma 20 implies $[R, H(R)] \subseteq J(R)$. Hence $[\pi_\alpha(R), \pi_\alpha(H(R))] \subseteq \pi_\alpha(J(R))=0$. Since $\pi_\alpha(R)=R_\alpha$, $\pi_\alpha(H(R)) \subseteq Z(R_\alpha)$ as desired.

3. THE PROOF OF THEOREM 2

Proof of Theorem 2. Since $N(R)=0$, R is a subdirect product of prime rings R_α with $N(R_\alpha)=0$. Since the image of $T_k(R)$ under the natural homomorphism from R onto R_α is included in $T_k(R_\alpha)$, it suffices to prove that $T_k(R_\alpha) \subseteq Z(R_\alpha)$ for each α.

Hence, without loss of generality, we may assume that R is a prime ring with $N(R)=0$. If char $R=p>0$ or $N_r(R)=0$, then the assertion follows immediately from Lemma 11 or from Theorem 4, respectively. So we may assume henceforth that char $R=0$ and $N_r(R)\neq 0$.

Let $a\in T_k(R)$ and let $b\in R$ be such that $b^t\neq 0$ but $b^{t+1}=0$ for some $t\geqslant 1$. Since $T_k(R)$ is obviously invariant under any automorphisms of R, we have $(1-\lambda b)\,a(1-\lambda b)^{-1}\in T_k(R)$ for $\lambda=1, 2, ..., t$. Now compute

$$\begin{aligned}(1-\lambda b)\,a(1-\lambda b)^{-1} &= (1-\lambda b)\,a(1+(\lambda b)+(\lambda b)^2+\cdots+(\lambda b)^t)\\ &= a+\lambda(ab-ba)+\lambda^2(ab-ba)b+\cdots\\ &\quad +\lambda^{t+1}(-ba)b^t\in T_k(R).\end{aligned}$$

Using the Vandermonde determinant argument, we can solve $\mu[a,b]\in T_k(R)$ for some nonzero integer μ. Since char $R=0$, $[a,b]\in T_k(R)$. Since each element of $N_r(R)$ is a sum of nilpotent elements, we have thus shown that $[T_k(R), N_r(R)]\subseteq T_k(R)$. By Theorem 5 of [9], either $[T_k(R), N_r(R)]=0$ or there exists a nonzero ideal M of R such that $[M, R]\subseteq T_k(R)$. If $[T_k(R), N_r(R)]=0$, then $T_k(R)\subseteq Z(R)$ by the primeness of R. So assume that $T_k(R)\supseteq [M, R]$. Given x, y in M, $[x,y]\in T_k(R)$. Hence there exists $n=n(x,y)\geqslant 1$ such that $[[x,y], y^n]_k=0$, and so $[x, y^n]_{k+1}=0$. Thus M is a C_{k+1}-ring. By Theorem 1, M is commutative and so is R. Thus $T_k(R)\subseteq Z(R)$.

The following immediate consequence is a special instance of Conjecture C:

Corollary 2. *Assume that R satisfies the following condition: For given x, y in R, there exist $m=m(x)\geqslant 1$, $n=n(x,y)\geqslant 1$, and $k=k(x)\geqslant 1$ such that $[x^m, y^n]_k=0$. If $N(R)=0$, then R is commutative.*

Proof. Let $x\in R$. By the condition assumed on R, there exist positive integers $m=m(x)$ and $k=k(x)$ such that $x^m\in T_k(R)$. Since $T_k(R)=Z(R)$ by Theorem 2, $x^m\in Z(R)$. Now our theorem follows from Theorem 3.2.2 of [8].

4. The Proof of Theorem 3

Lemma 23. *Assume that R is a C-ring. If $N_r(R)=0$ then R is reduced.*

Proof. Let $a\in R$ be such that $a^2=0$. For $y\in R$, there exist positive integers $m=m(a,y)\geqslant 1$, $n=n(a,y)\geqslant 1$, and $k=k(a,y)\geqslant 1$ such that $[(a+ay)^m, (ay)^n]_k=0$. Since $(a+ay)^m=(ay)^{m-1}a+(ay)^m$, we have $0=[(a+ay)^m, (ay)^n]_k = [(ay)^{m-1}a+(ay)^m, (ay)^n]_k = [(ay)^{m-1}a, (ay)^n]_k =$

$(-1)^k (ay)^{nk+m-1} a$. Hence aR is nil. Since $N_r(R)=0$, $a=0$. So R is reduced.

By the result of [2], a reduced ring is a subdirect product of domains. We may assume henceforth that R is a domain. For brevity, we call R a *C-domain* if R is a domain satisfying the condition (C). The following lemma, whose proof is essentially the same as that of Lemma 1, is also well known.

LEMMA 24. *Assume that R is a C-ring and $N(R)=0$. If R is torsion, then R is commutative.*

The hardest case of Theorem 2 is when R is a domain of characteristic 0. Drazin [4] proved the case when R is a division ring. Our proof below does not depend on his result and works for any domains. The following is crucial,

LEMMA 25. *Assume that R is a C-domain of characteristic* 0. *If R has the unit element* 1, *then R is commutative.*

Proof. We may assume that R is an algebra over the field of rational numbers. For each $y \in R$, consider the set $R_y = \{x \mid x \in R \text{ and } [x, y^n]_k = 0$ for some $n, k \geqslant 1\}$. It is obvious that R_y forms a subring of R. By the condition (C) on R, R is radical over R_y. Then, by Theorem 2 of [3], $R_y = R$. Hence for each x, y in R, there exist $n, k \geqslant 1$ such that $[x, y^n]_k = 0$. So $R \subseteq H(R) \subseteq Z(R)$ by Theorem 4. Thus R is commutative.

Let R be a C-domain of characteristic 0. For each nonzero element y in R, we recall that $C(y) = \{x \in R \mid xy = yx\}$ and $W(y) = \{x \in R \mid xy^t = y^t x$ for some $t \geqslant 1\}$.

LEMMA 26. *Assume that R is a C-domain of characteristic* 0. *Then $C(y)$ is commutative.*

Proof. Let S be the localization of $C(y)$ at the semigroup generated by y. Then S is also a C-domain of characteristic 0. Now S possesses the unit element 1. By Lemma 25, S and hence $C(y)$ also must be commutative.

LEMMA 27. *Assume that R is a C-domain of characteristic* 0. *Then $W(y)$ is commutative and hence $W(y) = C(y)$.*

Proof. Let $x, z \in W(y)$. Pick $t \geqslant 1$ such that $x, z \in C(y^t)$. By Lemma 26, $xz = zx$. So $W(y)$ is commutative. Since $y \in W(y)$, $[W(y), y] = 0$, so $W(y) \subseteq C(y)$. Thus $W(y) = C(y)$.

LEMMA 28. *Assume that R is a C-domain of characteristic* 0. *Let $x, y \in R$. If $[x^m, y^n]_k = 0$ for some $m, n, k \geqslant 1$ then $[x^m, y]_k = 0$.*

Proof. We may assume that $y \neq 0$. We proceed by induction on k. If $k=1$, then $x^m \in W(y) \subseteq C(y)$ by Lemma 27, and hence $[x^m, y]=0$. Assume that the lemma is true for k and suppose that $[x^m, y^n]_{k+1}=0$. Then $[x^m, y^n]_k \in W(y) \subseteq C(y)$. So $[[x^m, y^n]_k, y]=0$. Hence $[[x^m, y], y^n]_k = [[x^m, y^n]_k, y]=0$. By induction hypothesis, $[[x^m, y], y]_k=0$. So $[x^m, y]_{k+1}=0$ as desired. This completes the induction step.

LEMMA 29. *Assume that R is a C-domain of characteristic* 0. *If $[a, b] \in C(b)$ then for any $m, k \geqslant 1$, $[b^m, a]_k \in C(b)$.*

Proof. Induction on k. Let $a, b \in R$ be such that $[a, b] \in C(b)$. We may assume that $b \neq 0$. The case that $k=1$ is obvious. Now, assuming that $[b^m, a]_k \in C(b)$ as the induction hypothesis, we show that $[b^m, a]_{k+1} \in C(b)$. Set $c=[b^m, a]_k$. Then $[c, b]=0$. Hence $[[c, b], a]=0$. Using Jacobi identity, $0=[[c, b], a]=[[c, a], b]+[c, [b, a]]$. Since both c and $[b, a]$ are in $C(b)$, $[c, [b, a]]=0$ by Lemma 26. So $[[c, a], b]=0$. Hence $[b^m, a]_{k+1}=[c, a] \in C(b)$ as desired.

LEMMA 30. *Assume that R is a C-domain of characteristic* 0. *If $[a, b] \in C(b)$, then $[a, b]=0$.*

Proof. By the condition (C), there exist $m, n, k \geqslant 1$ such that $[b^m, a^n]_k=0$. By Lemma 28, $[b^m, a]_k=0$. If $k=1$, then $a \in W(b) \subseteq C(b)$ and hence $[a, b]=0$ as desired. So we assume that $k>1$. We may also assume that $[b^m, a]_{k-1} \neq 0$, for otherwise we can replace k by $k-1$. By Lemma 29, $[b^m, a]_{k-1} \in C(b)$. Hence $b \in C([b^m, a]_{k-1})$. Since $a \in C([b^m, a]_{k-1})$, we have $[a, b]=0$ by the commutativity of $C([b^m, a]_{k-1})$.

LEMMA 31. *Assume that R is a C-domain of characteristic* 0. *Then R is commutative.*

Proof. Given $x, y \in R$, there exist $m, n, k \geqslant 1$ such that $[x^m, y^n]_k=0$. If $k=1$, then $[x, y]=0$ by Lemma 28. If $k=2$, then $[x^m, y^n] \in C(y^n)$ and hence $[x^m, y^n]=0$ by Lemma 30. So $[x, y]=0$ as above. If $k>2$, set $[x^m, y^n]_{k-2}=a$ and $y^n=b$. Then $[[a, b], b]=[x^m, y^n]_k=0$. So $[a, b] \in C(b)$. By Lemma 30, $0=[a, b]=[x^m, y^n]_{k-1}$. Repeating the same argument in this manner, we can finally show that $[x^m, y^n]=0$. So $[x, y]=0$, as desired.

Theorem 3 is thus proved.

5. THE PROOF OF THEOREM 5

The aim of this section is to prove that both Conjecture C and Conjecture H are equivalent to the weaker Conjecture K.

Proof of Theorem 5. We prove the equivalence by showing that Conjecture $C \Rightarrow$ Conjecture $K \Rightarrow$ Conjecture $H \Rightarrow$ Conjecture C.

Conjecture $C \Rightarrow$ Conjecture K: This is obvious.

Conjecture $K \Rightarrow$ Conjecture H: Suppose that R is a ring such that $N(R)=0$. Using the embedding in Lemma 2.2.3 of [8], we may assume that R is prime. If $N_r(R)=0$, then $H(R)=Z(R)$ by Theorem 4. So we assume that $N_r(R)\neq 0$. Since $H(R)$ is obviously a subring of R which is invariant under any automorphisms of R, by the main result of [13], either $H(R)\subseteq Z(R)$ or $H(R)$ contains a nonzero ideal I of R. If $H(R)\subseteq Z(R)$, then we are done. So assume that $H(R)$ contains a nonzero ideal I of R. But I is obviously a K-ring and, by Conjecture K, I is commutative. Hence R is also commutative and trivially $H(R)=Z(R)$.

Conjecture $H \Rightarrow$ Conjecture C: Let R be a C-ring such that $N(R)=0$. As before, without loss of generality, we may assume that R is prime. By Lemma 24, we may further assume that R is of characteristic 0. Suppose that $a\in R$ is such that $a^2=0$. Given $x\in R$, by the Condition (C) on R, there exist positive integers $m_1, n_1, k_1, m_2, n_2, k_2 \geqslant 1$ such that

$$[(1+a)\,x^{m_1}(1-a), x^{n_1}]_{k_1}=0$$

and

$$[(1-a)\,x^{m_2}(1-a), x^{n_2}]_{k_2}=0.$$

Set $m=m_1m_2$, $n=n_1n_2$, and $k=m_2k_1+m_1k_2$. Then

$$[(1+a)\,x^{m}(1-a), x^{n}]_{k}=0=[(1-a)\,x^{m}(1+a), x^{n}]_{k}$$

so $2[ax^m - x^m a, x^n]_k = [(1+a)\,x^m(1-a) - (1-a)\,x^m(1+a), x^n]_k = 0$. Since char $R=0$, $[[a, x^m], x^n]_k=0$. So $a\in H(R)\subseteq Z(R)$ by Conjecture H. Since R is prime, $Z(R)$ does not contain any nonzero nilpotent elements, so $a=0$. Thus R is reduced. By Theorem 3, R is commutative, as desired.

Appendix

As there is no convenient reference known to us, we include a proof of the following simple fact for the sake of completeness:

PROPOSITION. *Let R be a torsion-free ring and let $P(R)$ be its prime radical. Then the quotient ring $R/P(R)$ is also torsion-free.*

Proof. An infinite sequence $M=\{a_0, a_1, ..., a_i, ...\}$ is called an m-sequence beginning with a if $a_0=a$ and $a_{i+1}\in a_iRa_i$ for all $i\geqslant 0$. For

$a \in R$, it is well known that $a \in P(R)$ if and only if any m-sequence beginning with a contains the zero element 0. (See, for example, Neal H. McCoy, "The Theory of Rings," Macmillan Co., New York, 1964.)

Let $a \in R$ be such that $na \in P(R)$ for some nonzero integer n. In order to show that $a \in P(R)$, it suffices to show that any m-sequence beginning with a must contain 0. So let $M = \{a = a_0, a_1, ...\}$ be an arbitrary m-sequence beginning with a. For each $i \geqslant 0$, since $a_{i+1} \in a_i R a_i$, $n^{2^{i+1}} a_{i+1} \in (n^{2^i} a_i) R (n^{2^i} a_i)$. Hence $M' = \{na, n^2 a_1, n^4 a_2, ..., n^{2^i} a_i, ...\}$ is an m-sequence beginning with na. Since $na \in P(R)$, some element of M' must be zero. Let us say $n^{2^i} a_i = 0$ for some $i \geqslant 0$. Since R is torsion-free, $a_i = 0$. Thus the m-sequence M also contains the zero element as desired.

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